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A remark on Lin and Chang's paper 'Consistent modeling of S&P 500 and VIX derivatives'*

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Abstract

Lin and Chang (2009, 2010) establish a VIX futures and option pricing theory when modeling S&P 500 index by using a stochastic volatility process with asset return and volatility jumps. In this note, we prove that Lin and Chang's formula is not an exact solution of their pricing equation. More generally, we show that the characteristic function of their pricing equation cannot be exponentially affine, as proposed by them. Furthermore, their formula cannot serve as a reasonable approximation. Using the Heston (1993) model as a special case, we demonstrate that Lin and Chang formula misprices VIX futures and options in general and the error can become substantially large.

Keywords: VIX option pricing, Affine Jump Diffusion, Characteristic Function.

JEL classification: G13.

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1 Introduction

VIX options have become very successful exchange-listed products for volatility trading. The bid-ask spread of VIX options market is large due to the fact that a commonly accepted VIX option pricing model is not available yet. Hence, developing a tractable VIX option pricing model is important for the healthy growth of the new market. Yet, as the VIX index is directly linked to the implied volatility of the S&P 500 index and hence to index options, a VIX option pricing model needs to provide enough flexibility to jointly price in a consistent manner options on the S&P 500 as well as on the VIX index.

The first attempt to express the price of VIX futures was made in Zhang and Zhu (2006), where the stochastic volatility model of Heston (1993) is used to describe S&P 500. They developed a simple theoretical model for VIX futures prices and tested the model using the actual futures price on one particular day. Dotsis, Psychoyios, and Skiadopoulos (2007) studied the continuous-time models of the volatility indices. Zhu and Zhu and Zhang (2007) further derived a no-arbitrage pricing model for VIX futures using the time-dependent long-term mean level in the volatility model. Lin (2007) incorporates simultaneous jumps in both asset return and volatility processes. Sepp (2008) used the square root stochastic variance model with jumps in the variance process to describe the evolution of S&P500 volatility, and showed how to price and hedge VIX futures and VIX options in this model. Albanese, Lo, and Mijatović (2009) studied volatility derivatives by using spectral methods. Zhang and Huang (2010) studied the CBOE S&P500 three-month variance futures market, and showed a linear dependence between the price of fixed time-to-maturity variance futures and the VIX by using a simple mean-reverting stochastic model for the S&P500 index. Lu and Zhu (2010) studied the variance term structure using VIX futures market. Zhang, Shu, and Brenner (2010) studied VIX futures market by using a stochastic volatility model with stochastic long-term mean level. Some other recent studies about the VIX and its derivatives include Chen, Chung, and Ho (2010), Dupoyet, Daigler, and Chen (2011), Hilal, Poon, and Tawn (2011), Konstantinidi and Skiadopoulos (2011), Cont and Kokholm (2011), Shu and Zhang (2012), and Zhu and Lian (2012) among others. Carr and Lee (2009) provided an interesting review on volatility derivatives market.

Lin and Chang (2009, 2010) establish a VIX futures and option pricing theory when modeling

S&P 500 index by using a stochastic volatility process with asset return and volatility jumps. Hence, their model seems to suggest a pricing framework which is both tractable and flexible enough to consistently price index options and options on the VIX. However, we show that Lin and Chang's (2009, 2010) formula published in both papers is not an exact solution of their pricing equation. More generally, we formally prove that the characteristic function of their pricing equation cannot be exponentially affine, as proposed by them. One could still argue that their formula provides a reasonable approximation for an option pricing formula that, given their general setup, does not allow for a closed-form solution. However, by using a reduced-form specification of their model, we find that their formula can also not serve as an approximation. In particular, we use the simple setup of the Heston (1993) stochastic volatility model and we demonstrate that Lin and Chang formula misprices VIX futures and options in general and the error could be substantially large. We further point out that for the simultaneous pricing of index and VIX options, an exact formula has actually been provided by Sepp (2008) under the assumption of a stochastic volatility process with volatility jumps but no jumps in asset return. In the more general case of jumps in both price and volatility processes and with an additional long-term volatility factor, the pricing formulas have been established in Bardgett, Gourier, and Leippold (2011).

This note is structured as follows. In the next section, we briefly review some general results on affine jump diffusions and their characteristic function. In Section 3, we present the main result of Lin and Chang (2009, 2010). In Section 4, we provide a formal proof showing that the result of Lin and Chang cannot be correct and we also show that their formula cannot serve as an appropriate approximation of the true pricing formula. Section 5 concludes.

2 Affine Jump Diffusion

Let $\mathcal{X} \subset \mathbb{R}$ be a closed set with non-empty interior. Throughout this note we assume that for every $x \in \mathcal{X}$ there exists a solution $X = X^x$ of the one-dimensional stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + dJ_t, \quad X(0) = x, \tag{1}$$

where J is a pure-jump process with jump arrival intensity $\Lambda(S_t)$ at time t for some $\Lambda : \mathbb{R} \to [0, \infty)$. Jump sizes $Z_1, Z_2, \dots$ are *iid* and independent of the Brownian motion B, which is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Definition 1: We call X *affine* if the $\mathcal{F}_t$-conditional characteristic function of X_T is exponential affine in X_t, for all $t \leq T$. That is, there exist $\mathbb{C}$-valued functions $\phi(T-t, z)$ and $\psi(T-t, z)$ with jointly continuous t-derivatives such that $X = X^x$ satisfies

$$\mathbb{E}[e^{zX_T} \mid \mathcal{F}_t] = \mathbb{E}_t[e^{zX_T}] = e^{\phi(T-t,z)+\psi(T-t,z)X_t}, \tag{2}$$

for all $z \in i\mathbb{R}, t \leq T$ and $x \in \mathcal{X}$.

Before we explain why the calculations of Lin and Chang are wrong, we briefly elaborate on an example in which the problem of determining a characteristic function is reduced to solving a system of ordinary differential equations (ODE). The same strategy is followed by Lin and Chang (2010) to find a solution for the characteristic function of the logarithm of the VIX squared and therefore deserves some attention.

Example 1: We consider the calculation of the following expectation

$$f(X_t, t) = \mathbb{E}(e^{X_T} \mid X_t) \tag{3}$$

under the assumption of affine dependence of μ and σ^2 on X, i.e., we assume $\mu(x) = a + bx, \sigma(x)^2 = cx$ and $\lambda(x) = l_o + l_1 x$ for some coefficients $a, b, c, l_0, l_1 \in \mathbb{R}$. If f has two continuous derivatives, the application of Itô's formula for jump diffusions gives

$$\begin{aligned} f(X_t, t) =& f(X_0, t) + \int_0^t \gamma(X_{s-}, s) ds + \int_0^t f_x(X_{s-}, s) dB_s \\ &+ \sum_{0<s\leq t} [f(X_s, s) - f(X_{s-}, s)], \end{aligned} \tag{4}$$

where

$$\gamma(x, t) = f_t(x, t) + f_x(x, t)\mu(x) + \frac{1}{2} f_{xx}(x, t)\sigma(x)^2. \tag{5}$$

Under some technical regularity conditions we can show that $f(X_t, t)$ is a martingale. Therefore, we get

$$
\begin{aligned}
0 = & f_t(x,t) + f_x(x,t)\mu(x) + \frac{1}{2} f_{xx}(x,t)\sigma(x)^2 \\
& + \mathbb{E}[\lambda(x+Z_i) f(x+Z_i,t) - \lambda(x) f(x,t)].
\end{aligned} \tag{6}
$$

To solve the above partial differential equation (PDE) we conjecture a solution of the form $f(x,t) = e^{\alpha(T-t)+\beta(T-t)x}$. Substituting this conjectured solution into (6) we obtain

$$
\begin{aligned}
& e^{\alpha(T-t)+\beta(T-t)x}\big(-\alpha'(s) - \beta'(s)x + \beta(s)(a+bx) \\
& + \frac{1}{2}\beta(s)^2 c^2 x + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1(\mathbb{E}[Z_i e^{\beta(s)Z_i}] + \mathbb{E}[(e^{\beta(s)Z_i} - 1)]x\big) = 0.
\end{aligned} \tag{7}
$$

Dividing by $e^{\alpha(T-t)+\beta(T-t)x}$ and collecting terms in x, we get

$$
u(s)x + v(s) = 0, \tag{8}
$$

where

$$
\begin{aligned}
u(s) &= -\beta'(s) + \beta(s)b + \frac{1}{2}\beta(s)^2 + l_1[\mathbb{E}(e^{\beta(s)Z_i}) - 1)] \\
v(s) &= -\alpha'(s) + \beta(s)a + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1 \mathbb{E}[Z_i e^{\beta(s)Z_i}].
\end{aligned} \tag{9}
$$

Because (8) must hold for all x, we have $u(s) = v(s) = 0$ for all $s \in \mathbb{R}$. Therefore, we can reduce the PDE to a set of ODE's, namely:

$$
\begin{aligned}
\beta'(s) &= \beta(s)b + \frac{1}{2}\beta(s)^2 + l_1[\mathbb{E}(e^{\beta(s)Z_i}) - 1)] \\
\alpha'(s) &= \beta(s)a + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1 \mathbb{E}[Z_i e^{\beta(s)Z_i}].
\end{aligned} \tag{10}
$$

Solving this system of ODE's leads to a solution of the PDE and therefore to a solution for (3), i.e., for the characteristic function of X.

Theorem 1: *Let $X = X^x$ be the solution of the stochastic differential equation defined in* (1) *with initial condition $X_0 = x$ for all $x \in \mathcal{X}$ for some closed subset $\mathcal{X} \subset \mathbb{R}$. Assume that X is affine as in Definition 1. Further, assume that the jump intensity λ is affine in X. Then the drift and the variance have affine dependence on the current state X_s.*

Proof. See Appendix A.

Remark 1: To simplify the proof of Theorem 1 and as it is enough for our purpose, we assume an affine jump intensity λ. For a more general result, we refer to Duffie, Filipovic, and Schachermayer (2003), Theorem 2.12.

3 A review of Lin and Chang's results

In Lin and Chang's model, the forward price of the S&P 500 index, denoted as F_t^T, is modeled as a jump-diffusion process with stochastic instantaneous variance v_t. Under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$, these processes are defined as

$$d \ln F_t^T = -\frac{1}{2} v_t dt + \sqrt{v_t} d\omega_{S,t} + z_S dN_t - \kappa \lambda_t dt, \tag{11}$$

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v \sqrt{v_t} d\omega_{v,t} + z_v dN_t, \tag{12}$$

where $\omega_{S,t}$ and $\omega_{v,t}$ are two $\mathbb{Q}$-Brownian motions with correlation coefficient ρ. Asset returns and variance jump at the same time according to the poisson process N_t. The variance jump size z_v is exponentially distributed with mean $\mu_v > 0$, i.e., its probability density is given by $p(z_v) = \frac{1}{\mu_v} e^{-\frac{z_v}{\mu_v}}$, $0 \leq z_v < +\infty$. To introduce correlated jump sizes, the asset return jump size z_S is conditioned on the realization of z_v. In particular, z_S is normally distributed with mean $\mu_j + \rho_j z_v$ and variance σ_j^2. The jump intensity is assumed to be $\lambda_t = \lambda_0 + \lambda_1 v_t$ and the relative forward price jump size, $J \equiv e^{z_S} - 1$, has a mean given by[1]

$$\kappa \equiv \mathbb{E}^{\mathbb{Q}}(e^{z_S} - 1) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(e^{z_S}|z_v)] - 1 = \mathbb{E}^{\mathbb{Q}}\left(e^{\mu_j + \rho_j z_v + \frac{1}{2}\sigma_j^2}\right) - 1 = \frac{e^{\mu_j + \frac{1}{2}\sigma_j^2}}{1 - \rho_j \mu_v} - 1.$$

[1]It seems to us that the notation J_t, frequently used in the literature including Lin and Chang, is not appropriate because J is a random number instead of a process.

The variance and covariance of the two jump sizes, z_v and z_S, are given by

$$\begin{aligned}
\mathrm{Var}(z_v) &= \mathbb{E}^{\mathbb{Q}}[(z_v-\mu_v)^2] = \mathbb{E}^{\mathbb{Q}}(z_v^2) - \mu_v^2 = \mu_v^2, \\
\mathrm{Var}(z_S) &= \mathbb{E}^{\mathbb{Q}}[(z_S-\mu_j-\rho_j\mu_v)^2] = \mathbb{E}^{\mathbb{Q}}\{[(z_S-\mu_j-\rho_j z_v)+\rho_j(z_v-\mu_v)]^2\} = \sigma_j^2+\rho_j^2\mu_v^2, \\
\mathrm{Cov}(z_S, z_v) &= \mathbb{E}^{\mathbb{Q}}[(z_S-\mu_j-\rho_j\mu_v)(z_v-\mu_v)] = \mathbb{E}^{\mathbb{Q}}[\rho_j(z_v-\mu_v)^2] = \rho_j\mu_v^2,
\end{aligned}$$

hence the correlation coefficient between z_S and z_v is given by

$$\frac{\mathrm{Cov}(z_S, z_v)}{\sqrt{\mathrm{Var}(z_S)\cdot\mathrm{Var}(z_v)}} = \frac{\rho_j\mu_v}{\sqrt{\sigma_j^2+\rho_j^2\mu_v^2}}.$$

The variance process can be rewritten as

$$dv_t = \kappa_v^*(\theta_v^*-v_t)dt + \sigma_v\sqrt{v_t}d\omega_{v,t} + z_v dN_t - (\lambda_0+\lambda_1 v_t)\mu_v dt,$$

where

$$\kappa_v^* = \kappa_v - \lambda_1\mu_v, \qquad \theta_v^* = \frac{\kappa_v\theta_v+\lambda_0\mu_v}{\kappa_v-\lambda_1\mu_v},$$

are the effective mean-reverting speed and long-term mean level under the risk-neutral measure $\mathbb{Q}$. Note that the mean of jump process, $\mathbb{E}^{\mathbb{Q}}(z_v dN_t) = (\lambda_0+\lambda_1 v_t)\mu_v dt$, affects the parameter values of the mean-reversion process. Based on the CBOE definition, the VIX squared can be derived from[2]

$$\begin{aligned}
\mathrm{VIX}_t^2 &\equiv \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[-\ln\frac{S_{t+\tau}}{F_t^{t+\tau}}\right] = \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau}\frac{dS_t}{S_t} - d(\ln S_t)\right], \\
&= \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau}\frac{dF_t^T}{F_t^T} - d(\ln F_t^T)\right], \qquad F_t^T = S_t e^{r(T-t)}, \\
&= \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left\{\int_t^{t+\tau}\left[\frac{1}{2}v_t + (e^{z_S}-1-z_S)(\lambda_0+\lambda_1 v_t)\right]dt\right\} \\
&= \frac{\zeta_1}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left(\int_t^{t+\tau} v_t dt\right) + \zeta_2 = \frac{\zeta_1}{\tau}(a_\tau v_t + b_\tau) + \zeta_2,
\end{aligned} \tag{13}$$

[2] The result here is the same as Lin and Chang's, but the derivation is slightly different from that of Lin and Chang, in which they introduce an approximation on $\ln(1+J)$, which seems to be unnecessary at least in our view.

where $\tau = 30/365$ and

$$\zeta_1 = 1 + 2\lambda_1[\kappa - (\mu_j + \rho_j \mu_v)], \qquad \zeta_2 = 2\lambda_0[\kappa - (\mu_j + \rho_j \mu_v)],$$
$$a_\tau = \frac{1 - e^{\kappa_v^* \tau}}{\kappa_v^*}, \qquad b_\tau = \theta_v^*(\tau - a_\tau).$$

Denoting $L = \ln S$, the price of a European call option $C(\tau_C, L, v)$ written on VIX with the strike price K and time-to-maturity $\tau_C \equiv T - t$ satisfies the following integro-partial differential equation (IPDE)[3]

$$\begin{aligned}
&\frac{1}{2}v\frac{\partial^2 C}{\partial L^2} + \left[r - \lambda_0\kappa - \left(\lambda_1\kappa + \frac{1}{2}\right)v\right]\frac{\partial C}{\partial L} + \rho\sigma_v v\frac{\partial^2 C}{\partial L \partial v} \\
&+\frac{1}{2}\sigma_v^2 v\frac{\partial^2 C}{\partial v^2} + \kappa_v(\theta_v - v)\frac{\partial C}{\partial v} - \frac{\partial C}{\partial \tau_C} - rC \\
&+\mathbb{E}_t^{\mathbb{Q}}\{[\lambda_0 + \lambda_1(v + z_v)]C(\tau_C, L + z_S, v + z_v) - (\lambda_0 + \lambda_1 v)C(\tau_C, L, v)\} = 0,
\end{aligned}$$

with final condition $C(\tau_C = 0, L, v) = \max(\mathrm{VIX}_T - K, 0)$, where $\mathrm{VIX}_T = \sqrt{\zeta_1 a_\tau v_T/\tau + \zeta_1 b_\tau/\tau + \zeta_2}$.

Lin and Chang claim that they have obtained a closed-form VIX option pricing formula as follows

$$C(\tau_C, L, v) = F_t^{VIX}(T)e^{-r\tau_C}\Pi_1 - Ke^{-r\tau_C}\Pi_2, \tag{14}$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi \ln K^2} f_2(\tau_C; i\phi + 1/2)}{i\phi f_2(\tau_C; 1/2)}\right] d\phi, \tag{15}$$
$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi \ln K^2} f_2(\tau_C; i\phi)}{i\phi}\right] d\phi, \tag{16}$$

and $f_2(\tau_C; i\phi) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi \ln \mathrm{VIX}_T^2}\right]$, the characteristic function of $\ln \mathrm{VIX}_T^2$ given by

$$f_2(\tau_C; i\phi) = \exp[C_2(\tau_C) + J_2(\tau_C) + D_2(\tau_C)\ln \mathrm{VIX}_t^2], \tag{17}$$

where $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ are defined in equation (B.10) in Lin and Chang (2010).

[3] In Lin and Chang, both variables t and τ_C are used as independent variables in the option price function, $C(t, \tau_C)$. Here we choose to use one of them, τ_C, as they are related by $\tau_C \equiv T - t$.

4 Disproving the correctness of Lin and Chang's formula

4.1 Formal Argument

We start by presenting the following result, which is based on a formal argument outlined in the appendix.

PROPOSITION 1: *Lin and Chang's formula (14, 15, 16, 17) with $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ given by their equations (B.10) in Lin and Chang (2010) is not an exact solution of their pricing equation (14).*

Proof. See Appendix B.

We first note that using equations (14, 15, 16), Lin and Chang describe VIX option price in terms of the characteristic function of $\ln \mathrm{VIX}_T^2$, i.e., $f_2(\tau_C; i\phi)$. This representation is fine because it is consistent with Bakshi and Madan (2000). The key issue here is the analytical tractability of $f_2(\tau_C; i\phi)$, without which the representation does not help us much in computing the VIX option prices.

When Lin and Chang solve the problem, they conjecture in their equation (B.4) that the characteristic function of $\ln \mathrm{VIX}_T^2$ has the following form:

$$f_2(\tau_C; i\phi) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi \ln \mathrm{VIX}_T^2}\right] = e^{C_2(\tau_C)+J_2(\tau_C)+D_2(\tau_C)\ln \mathrm{VIX}_t^2+G_2(\tau_C)L_t}. \tag{18}$$

By imposing such a structure, they implicitly assume that $C_2(\tau_C)$, $J_2(\tau_C)$ and $D_2(\tau_C)$ are not functions of VIX_t when they derive ODEs for them. However, in the final result of their equation (B.10), $C_2(\tau_C)$, $J_2(\tau_C)$, and $D_2(\tau_C)$ are indeed functions of VIX_t, which contradicts their original assumption. Therefore, their conjecture (18) cannot be appropriate.

We also note that during the process of solving for $f_2(\tau_C; i\phi)$, Lin and Chang (2010) introduce an approximation in their equation (B.6) for $\exp[i\phi \ln(1 + (\mu_v/\mathrm{VIX}_T^2))]$ by using Taylor's expansion at VIX_t^2. However, the error of this approximation is not analyzed.[4]

[4] Also, the variable M in equations below their equation (B.6) is never defined in Lin and Chang (2010).

What is the reason the method used by Lin and Chang (2010) fails? To give an answer to this question, we observe the following:

PROPOSITION 2: *The characteristic function of the stochastic process* $\ln(\mathrm{VIX}_t^2)$ *cannot be exponentially affine in* $\ln(\mathrm{VIX}_t^2)$.

Proof. See Appendix C.

The derivation of Proposition 2 in the appendix makes it obvious why the method used by Lin and Chang (2010) fails, namely because of non-affine dependence of the drift, variance and jump on $\ln(\mathrm{VIX}_t^2)$. A potential remedy to obtain at least a closed-form approximation for the characteristic function would be to apply a second-order perturbation of $\ln(\mathrm{VIX}_t^2)$ around some fixed volatility level. Such an approximation would lead to a characteristic function that is exponential linear-quadratic in VIX_t^2. However, in such a setting, additional care has to be applied to the specification of the volatility dynamics in a setting with jumps (see, e.g., Cheng and Scaillet (2007)).

4.2 Numerical Investigation

So far, we have presented a formal argument that Lin and Chang's formula for VIX option pricing cannot be correct. However, one might argue that their formula may produce reasonable prices and may therefore serve as an approximation of the true option pricing formula. Being an approximate formula for the prices of VIX options and futures, its accuracy is important for users. Unfortunately, with some numerical analysis, we find that in general, Lin and Chang's formula (14, 15, 16, 17) clearly misprices VIX options and futures. Furthermore, the error could be substantially large.

To substantiate our claim, we use a simplified case to analyze the error of Lin and Chang's formula. In particular, we use the classical Heston model for stochastic volatility (Heston (1993)). Under such a specification, the conditional risk-neutral probability density function of VIX_T, $f^{\mathbb{Q}}(\mathrm{VIX}_T|\mathrm{VIX}_t)$ has been provided by Zhang and Zhu (2006), which can be used to calculate the prices of VIX futures and options given by

$$\mathrm{VIXF}_t^T = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{VIX}_T], \tag{19}$$

$$C(T-t, L, v) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[\max(\mathrm{VIX}_T - K, 0)]. \tag{20}$$

Using the parameter values estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006), $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$, and we assume the current VIX level is at 15% and the riskfree rate is $r = 2\%$. The prices of VIX futures and options with different maturities are presented in Table 1 and 2.

[TABLE I about here]

As we can see from the Tables, Lin and Chang's formula (14, 15, 16, 17) misprices VIX options and futures. The error could be substantially large.[5] Lin and Chang formula overprices two-month VIX options by about 40%. The overpricing could be even higher than 100% for one-year VIX options. The overpricing for VIX futures is also large even though it is smaller than that for VIX options.

[TABLE II about here]

In equation (B.10) in Lin and Chang (2010), The variable B appears in $e^{B\tau_C}$, therefore $B\tau_C$ has to be dimensionless. However, from the formula for B, we can tell that it is not dimensionless due to the last term $1/\ln \mathrm{VIX}_t^2$. This indicates that the formula for B has some problems. Indeed, note that for VIX futures and options with a very long maturity, i.e., $T - t \to +\infty$, we have

$$\lim_{T-t\to+\infty} v_T = \theta_v^*,$$

and

$$\lim_{T-t\to+\infty} \mathrm{VIX}_T = \sqrt{\zeta_1\theta_v^* + \zeta_2}.$$

Then the VIX futures price has the same limit

$$\lim_{T-t\to+\infty} \mathrm{VIXF}_t^T = \sqrt{\zeta_1\theta_v^* + \zeta_2}, \tag{21}$$

[5] Note, the VIX options with a maturity of one to two months are the most liquid ones.

and the forward VIX call option price has the limit as follows

$$\lim_{T-t\to+\infty} e^{r(T-t)}C(T-t,L,v) = \max(\sqrt{\zeta_1\theta_v^* + \zeta_2} - K, 0). \tag{22}$$

The asymptotic behavior of Lin and Chang's formula, depending on the sign of the value of B, does not follow the property above in general.

5 Conclusion

In this note, we prove that Lin and Chang's (2009, 2010) formula is not an exact solution of their pricing equation. Using as a reduced specification the simple case of the Heston (1993) model, we demonstrate that Lin and Chang's formula misprices VIX futures and options in general and the error could be substantially large. We further point out that an exact formula has actually been provided by Sepp (2008) and for a more general setting by Bardgett, Gourier, and Leippold (2011).

The empirical features on VIX options market provided by Lin and Chang (2010) are based on their in-accurate formula. They need to be reexamined immediately by using the correct VIX option pricing formula. Other research that uses Lin and Chang's formula such as, e.g., Wang and Daigler (2011) and Chung, Tsai, Wang, and Weng (2011) and also needs to be reexamined.

A Proof of Theorem 1

Define the function

$$M(X_s, s) = e^{\phi(t-s,z)+\psi(t-s,z)X_s}. \tag{23}$$

Using Itô's formula as in (6) we obtain the equation

$$\begin{aligned} 0 =& M(X_s,s)\big(-\partial_t\phi(t-s,z) - \partial_t\psi(t-s,z)X_s + \psi(t-s,z)\mu(X_s) \\ &+ \frac{1}{2}\psi(t-s,z)^2\sigma(X_s)^2 + \mathbb{E}\left[\lambda(X_s+Z_i)M(Z_i,s) - \lambda(X_s)\right]\big) \end{aligned} \tag{24}$$

for all $s \le t$. Letting $s \to 0$ and dividing by $M(x,0)$, we thus obtain

$$\begin{aligned}
&\partial_t\phi(t,z) + \partial_t\psi(t,z)x \\
&= \psi(t,z)\mu(x) + \frac{1}{2}\psi(t,z)^2\sigma(x)^2 + \lambda_0\mathbb{E}[M(Z_i)-1] + \lambda_1\mathbb{E}[Z_i M(Z_i,0)] + \lambda_1\mathbb{E}[M(Z_i,0)-1]x
\end{aligned} \tag{25}$$

for all $x \in \mathcal{X}$ and $t \ge 0$, where we have written $\lambda(x) := \lambda_0 + \lambda_1 x$. Now since $\psi(0,z) = z$ we see that μ and σ^2 have to be affine in x.

B Proof of Proposition 1

Consider the special case of no-jump, i.e., $z_S = z_v = 0$, $\lambda_0 = \lambda_1 = 0$, hence $\kappa \equiv \mathbb{E}(e^{z_S} - 1) = 0$, $\kappa_v^* = \kappa_v$ and $\theta_v^* = \theta_v$. Then, the VIX formula simplifies to

$$\mathrm{VIX}_t^2 = \frac{1}{\tau}(a_\tau v_t + b_\tau),$$

where $a_\tau = \dfrac{1-e^{\kappa_v\tau}}{\kappa_v}$, $b_\tau = \theta_v(\tau - a_\tau)$. Note that $\zeta_1 = 1$ and $\zeta_2 = 0$. The VIX option pricing problem becomes

$$\begin{aligned}
&\frac{1}{2}v\frac{\partial^2 C}{\partial L^2} + \left(r - \frac{1}{2}v\right)\frac{\partial C}{\partial L} + \rho\sigma_v v\frac{\partial^2 C}{\partial L\partial v} \\
&\qquad + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 C}{\partial v^2} + \kappa_v(\theta_v - v)\frac{\partial C}{\partial v} - \frac{\partial C}{\partial \tau_C} - rC = 0, \\
&C(\tau_C = 0, L, v) = \max(\mathrm{VIX}_T - K, 0).
\end{aligned} \tag{26}$$

The Lin and Chang's formula becomes

$$C(\tau_C, L, v) = F_t^{VIX}(T)e^{-r\tau_C}\Pi_1 - Ke^{-r\tau_C}\Pi_2, \tag{27}$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2} f_2(\tau_C; i\phi + 1/2)}{i\phi f_2(\tau_C; 1/2)}\right] d\phi, \tag{28}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2} f_2(\tau_C; i\phi)}{i\phi}\right] d\phi, \tag{29}$$

and

$$\begin{aligned}
f_2(\tau_C; i\phi) &= \exp[C_2(\tau_C) + D_2(\tau_C)\ln \mathrm{VIX}_t^2], \qquad (30)\\
C_2(\tau_C) &= \frac{B}{A}\kappa_v\tau_C - \frac{\kappa_v}{A}\left\{B\tau_C - \ln\left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right]e^{B\tau_C}\right\}\right.\\
&\quad \left. + \ln\left[\left(i\phi + \frac{B}{A}\right)^{-1}\right]\right\},\\
D_2(\tau_C) &= -\frac{B}{A} + \left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right]e^{B\tau_C}\right\}^{-1},\\
A &= \frac{1}{2}\sigma_v^2\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau} - \frac{b_\tau}{a_\tau}\right)\left(\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\right)^2\left(\frac{1}{\ln\mathrm{VIX}_t^2}\right),\\
B &= \left[\kappa_v\theta_v - \frac{1}{2}\sigma_v^2\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau} - \frac{b_\tau}{a_\tau}\right) + \kappa_v\frac{b_\tau}{a_\tau}\right]\left(\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\right)\left(\frac{1}{\ln\mathrm{VIX}_t^2}\right).
\end{aligned}$$

Because $f_2(\tau_C; i\phi) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi\ln\mathrm{VIX}_T^2}\right]$ is the characteristic function of $\ln\mathrm{VIX}_T^2$, it must be a solution of pricing PDE (26). However, by substituting equation (30) into equation (26), we can show that it is not a solution of (26). Therefore, Lin and Chang's formula (14, 15, 16, 17) with $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ given by their equations (B.10) in Lin and Chang (2010) is not an exact solution of their pricing equation (14).

C Proof of Proposition 2

Recall the equation

$$\mathrm{VIX}_t^2 = a\cdot\nu_t + b, \tag{31}$$

by (8) of Lin and Chang (2010), where $a, b \in \mathbb{R}$ are defined as in Lin and Chang. Equivalently, we can write

$$\ln(\mathrm{VIX}_t^2) = \ln(a \cdot \nu_t + b). \tag{32}$$

Using Itô's formula, equation (32) transforms to

$$\begin{aligned} d\ln(\mathrm{VIX}_t^2) =& \Big(\frac{a}{\mathrm{VIX}_t^2}\kappa_\nu(\theta_\nu - a^{-1}\mathrm{VIX}_t^2 + b) - \frac{1}{2}\frac{a^2\theta_\nu}{(\mathrm{VIX}_t^2)^2}(a^{-1}\mathrm{VIX}_t^2 - b)\Big)dt \\ &+ \frac{a}{\mathrm{VIX}_t^2}\sigma_\nu(\sqrt{a^{-1}\mathrm{VIX}_t^2 - b})d\omega_{\nu,t} \\ &+ (\ln(\mathrm{VIX}_t^2 + az_\nu + b) - \ln(\mathrm{VIX}_t^2))dN_t. \end{aligned} \tag{33}$$

Equation (33) shows that the drift, the variance and the jump intensity are not affine in $\ln(\mathrm{VIX}_t^2)$ and therefore, by Theorem 1 and Remark 1, the characteristic function of $\ln(\mathrm{VIX}_t^2)$ cannot be exponential affine in $\ln(\mathrm{VIX}_t^2)$.

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Table 1: The prices of VIX futures with different maturities. The parameter values of the Heston (1993) model are taken to be $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$ that are estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006). The current VIX level is $\text{VIX}_0 = 15$. LC is obtained by using Lin and Chang's (2010) formula. ZZ is obtained by using Zhang and Zhu (2006) formula. RE is the relative error between LC and ZZ, computed as $LC/ZZ - 1$.

Maturity (year)	LC	ZZ	RE (%)
0.0	15.00	15.00	0.0
0.1	18.13	17.60	3.0
0.2	20.87	19.09	9.3
0.3	23.20	19.95	16.3
0.4	25.16	20.46	23.0
0.5	26.79	20.77	29.0
0.6	28.13	20.96	34.2
0.7	29.23	21.07	38.7
0.8	30.13	21.14	42.5
0.9	30.87	21.18	45.7
1.0	31.47	21.20	48.4
1.1	31.95	21.22	50.5
1.2	32.35	21.23	52.3

Table 2: The prices of VIX call options with different maturities. The parameter values of the Heston's Heston (1993) model are taken to be $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$ that are estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006). The current VIX level is $\text{VIX}_0 = 15$ and riskfree rate is $r = 2\%$. LC is obtained by using Lin and Chang's (2010) formula. ZZ is obtained by using Zhang and Zhu (2006) approach. RE is the relative error between LC and ZZ, computed as $LC/ZZ - 1$.

Maturity (year)	LC	ZZ	RE (%)
0.0	0.00	0.00	0.0
0.1	4.03	3.17	27.1
0.2	6.69	4.51	48.5
0.3	8.90	5.29	68.4
0.4	10.73	5.75	86.7
0.5	12.23	6.02	103.2
0.6	13.47	6.18	117.9
0.7	14.47	6.27	130.7
0.8	15.28	6.32	141.7
0.9	15.94	6.35	151.0
1.0	16.46	6.36	158.8
1.1	16.88	6.36	165.3
1.2	17.21	6.36	170.0